

# FREQUENCY-DEPENDENT TIME DECAY OF SCHRÖDINGER FLOWS

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**ABSTRACT.** We show that the presence of negative eigenvalues in the spectrum of the angular component of an electromagnetic Schrödinger hamiltonian  $H$  generically produces a lack of the classical time-decay for the associated Schrödinger flow  $e^{-itH}$ . This is in contrast with the fact that dispersive estimates (Strichartz) still hold, in general, also in this case. We also observe an improvement of the decay for higher positive modes, showing that the time decay of the solution is due to the first nonzero term in the expansion of the initial datum as a series of eigenfunctions of a quantum harmonic oscillator with a singular potential. A completely analogous phenomenon is shown for the heat semigroup, as expected.

## 1. INTRODUCTION

In this manuscript, we follow a research started in [4, 5] concerning on time decay of  $L^p$ -norms of solutions to scaling invariant electromagnetic Schrödinger equations. In dimension  $N \geq 2$ , let us consider the hamiltonian

$$H = \left( -i\nabla + \frac{\mathbf{A}\left(\frac{x}{|x|}\right)}{|x|} \right)^2 + \frac{a\left(\frac{x}{|x|}\right)}{|x|^2},$$

where  $\mathbf{A} \in C^1(\mathbb{S}^{N-1}; \mathbb{R}^N)$  is transversal, i.e.

$$(1.1) \quad \mathbf{A}(\theta) \cdot \theta = 0 \quad \text{for all } \theta \in \mathbb{S}^{N-1},$$

and  $a \in L^\infty(\mathbb{S}^{N-1}; \mathbb{R})$ . Here and in the sequel, we always denote by  $r := |x|$ ,  $\theta = x/|x|$ , so that  $x = r\theta$ . Associated to  $H$ , we study the Cauchy-problem for the Schrödinger equation

$$(1.2) \quad \begin{cases} \partial_t u = -iHu \\ u(x, 0) = u_0(x) \in L^2(\mathbb{R}^N), \end{cases}$$

with  $u = u(x, t) : \mathbb{R}^{N+1} \rightarrow \mathbb{C}$ .

A fundamental role in the description of the dynamics in (1.2) is played by the angular hamiltonian

$$(1.3) \quad L = \left( -i \nabla_{\mathbb{S}^{N-1}} + \mathbf{A} \right)^2 + a(\theta).$$

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*Date:* October 14, 2015.

*2000 Mathematics Subject Classification.* 35J10, 35L05.

*Key words and phrases.* Schrödinger equation, electromagnetic potentials, representation formulas, decay estimates.

The first author was supported by the project FIRB 2012: “Dispersive dynamics: Fourier Analysis and Variational Methods” funded by MIUR. .

Notice that  $L$  is a symmetric operator, with compact inverse. Therefore, no continuous and residual spectrum are present, and

$$\sigma(L) = \sigma_p(L) = \{\mu_1 \leq \mu_2 \leq \dots\} \subset \mathbb{R},$$

where the sequence  $\{\mu_k\}$  diverges and each eigenvalue has finite multiplicity (see [7, Lemma A.5]). For  $k \in \mathbb{N}$ ,  $k \geq 1$ , we denote by  $\psi_k$  the  $L^2(\mathbb{S}^{N-1}, \mathbb{C})$ -normalized eigenfunction of  $L$  corresponding to  $\mu_k$ , namely

$$(1.4) \quad \begin{cases} L\psi_k = \mu_k \psi_k(\theta), & \text{in } \mathbb{S}^{N-1}, \\ \int_{\mathbb{S}^{N-1}} |\psi_k(\theta)|^2 dS(\theta) = 1. \end{cases}$$

By repeating each eigenvalue as many times as its multiplicity, we can arrange the above enumeration in such a way that the correspondence  $k \leftrightarrow \psi_k$  is one-to-one. Hence, normalizing, we can construct the set  $\{\psi_k\}$  as an orthonormal basis in  $L^2(\mathbb{S}^{N-1}; \mathbb{C})$ .

The condition

$$(1.5) \quad \mu_1 > -\left(\frac{N-2}{2}\right)^2$$

implies that the quadratic form

$$q[\psi] := \int_{\mathbb{R}^N} \left| -i\nabla\psi + \frac{\mathbf{A}(x/|x|)}{|x|}\psi \right|^2 + \int_{\mathbb{R}^N} \frac{a(x/|x|)}{|x|^2} |\psi|^2,$$

associated to  $H$ , is positive (in dimension  $N = 2$  by definition, while in dimension  $N \geq 3$  by magnetic Hardy inequality, (see [15]). Therefore the hamiltonian  $H$  is realized as the self-adjoint extension (Friedrichs) of  $q$  on the natural form domain, and, by the Spectral Theorem, the hamiltonian flow  $e^{-itH}$  associated to equation (1.2) is well defined.

Many efforts have been spent in the last decades to understand the dispersive properties of  $e^{-itH}$ . In [4], Theorem 1.3, we stated a useful representation formula which reads as follows

$$(1.6) \quad u(x, t) = \frac{e^{\frac{i|x|^2}{4t}}}{i(2t)^{N/2}} \int_{\mathbb{R}^N} K\left(\frac{x}{\sqrt{2t}}, \frac{y}{\sqrt{2t}}\right) e^{i\frac{|y|^2}{4t}} u_0(y) dy,$$

provided (1.5) holds. Here we denote

$$(1.7) \quad K(x, y) = \sum_{k=1}^{\infty} i^{-\beta_k} j_{-\alpha_k}(|x||y|) \psi_k\left(\frac{x}{|x|}\right) \overline{\psi_k\left(\frac{y}{|y|}\right)},$$

where

$$(1.8) \quad \alpha_k := \frac{N-2}{2} - \sqrt{\left(\frac{N-2}{2}\right)^2 + \mu_k}, \quad \beta_k := \sqrt{\left(\frac{N-2}{2}\right)^2 + \mu_k},$$

and, for every  $\nu \in \mathbb{R}$ ,

$$j_\nu(r) := r^{-\frac{N-2}{2}} J_{\nu+\frac{N-2}{2}}(r)$$

with  $J_\nu$  denoting the Bessel function of the first kind

$$J_\nu(t) = \left(\frac{t}{2}\right)^\nu \sum_{k=0}^{\infty} \frac{(-1)^k}{\Gamma(k+1)\Gamma(k+\nu+1)} \left(\frac{t}{2}\right)^{2k}.$$

As an immediate consequence of (1.6), we have the following:

$$(1.9) \quad \sup_{x,y \in \mathbb{R}^N} |K(x,y)| < \infty \quad \Rightarrow \quad \|e^{-itH}\|_{L^1 \rightarrow L^\infty} \leq C|t|^{-\frac{N}{2}},$$

for some  $C > 0$  only depending on  $N$ , where  $\|e^{-itH}\|_{L^1 \rightarrow L^\infty}$  denotes the norm of  $e^{-itH}$  as an operator from  $L^1$  into  $L^\infty$ . Although proving the uniform boundedness of  $K$  can be hard, in [4, 5] we can do it in the following cases:

- if  $N = 2$ , for generic  $\mathbf{A}$ ,  $a$  in the above class;
- if  $N = 3$ ,  $\mathbf{A} \equiv 0$ , and  $0 \leq a$ -constant.

Starting from the dimension  $N = 3$ , the negative range

$$(1.10) \quad -\left(\frac{N-2}{2}\right)^2 < \mu_1 < 0$$

makes sense in this setting. Observe that, in the case  $a$  constant and  $\mathbf{A} \equiv 0$ , we have  $\mu_1 = a$ . In the case  $N = 3$  and  $\mathbf{A} \equiv 0$  we have already proved that if  $a \geq 0$  then the classical  $L^1$ - $L^\infty$  time decay in (1.9) holds; hence it is natural to wonder whether the  $L^1$ - $L^\infty$  time decay estimate still holds or not under condition (1.10). We stress that Strichartz estimates, which standardly follow by the  $L^1$ - $L^\infty$  bound, are known to hold in this case, as proved in [1, 2, 16]. Nevertheless, the best which is known about time-decay for the class of operators under consideration is in [4, 5], while for perturbative settings we refer to [17] as a standard reference.

The first aim of this paper is to give a negative answer to the above question, i.e., we want to show that condition (1.10) immediately destroys the time-decay of the free flow. We can now state our main result.

**Theorem 1.1.** *Let  $N \geq 3$ ,  $a \in L^\infty(\mathbb{S}^{N-1}, \mathbb{R})$ ,  $\mathbf{A} \in C^1(\mathbb{S}^{N-1}, \mathbb{R}^N)$ , and assume (1.1), (1.5), and (1.10). Then, for almost every  $t \in \mathbb{R}$ ,  $e^{-itH}(L^1) \not\subseteq L^\infty$ ; in particular  $e^{-itH}$  is not a bounded operator from  $L^1$  to  $L^\infty$ .*

**Remark 1.2.** Condition (1.10) needs to be read, eventually, in terms of the usual Hardy inequality, and its extension to the case of magnetic derivatives (see the standard reference [15]). Theorem 1.1 shows that condition  $\mu_1 \geq 0$  is necessary for the  $L^1 \rightarrow L^\infty$  bound. In other words, if the spherical Hamiltonian is not positive, the usual time decay property does not hold. As we see in the sequel, in the case  $\mu_1 \geq 0$  the rate of time decay (in suitable topologies) depends on the size of  $\mu_1$ .

The hint for the proof of Theorem 1.1 comes from estimate (1.29) in [4], which suggests the failure of the  $L^1$ - $L^\infty$  decay in the case of the inverse square potential. The proof of Theorem 1.1 shows that estimate (1.29) is sharp, and that the phenomenon is general, and related to the existence of the negative energy-level  $\mu_1 < 0$ . The key role in the proof is played by the operator

$$T := H + \frac{1}{4}|x|^2,$$

whose spectral properties are described in Section 2 below. Since  $T$  has discrete spectrum, and we can decompose  $L^2(\mathbb{R}^N)$  as a direct sum of eigenspaces for  $T$ , we can expand the initial datum  $u_0$  for (1.2) as a series of the following eigenfunctions of  $T$  forming an orthogonal basis of  $L^2(\mathbb{R}^N)$ :

$$(1.11) \quad V_{n,j}(x) = |x|^{-\alpha_j} e^{-\frac{|x|^2}{4}} P_{j,n}\left(\frac{|x|^2}{2}\right) \psi_j\left(\frac{x}{|x|}\right), \quad n, j \in \mathbb{N}, j \geq 1,$$

where  $P_{j,n}$  is the polynomial of degree  $n$  given by

$$P_{j,n}(t) = \sum_{i=0}^n \frac{(-n)_i}{\left(\frac{N}{2} - \alpha_j\right)_i} \frac{t^i}{i!},$$

denoting as  $(s)_i$ , for all  $s \in \mathbb{R}$ , the Pochhammer's symbol  $(s)_i = \prod_{j=0}^{i-1} (s+j)$ ,  $(s)_0 = 1$ . The main argument in the proof of Theorem 1.1 is that the evolution of those eigenfunctions, as initial data for (1.2), is quite explicit.

The second purpose of the present paper is to prove, when the classical time decay holds, e.g. in the case  $a \geq 0$  constant and  $\mathbf{A} \equiv 0$ , an improvement of the decay for higher positive modes. Roughly speaking, the more positive is  $\mu_1 > 0$ , the faster decay is expected to be, in suitable topologies (see e.g. [3, 6, 8, 11, 12, 13, 14] for some recent works related to this topic, both for Schrödinger and heat flows). For all  $k > 1$ , let us denote as

$$\mathcal{U}_k = \text{span} \{V_{n,j} : n \in \mathbb{N}, 1 \leq j < k\} \subset L^2(\mathbb{R}^N).$$

In the following theorem we observe that the time decay of the solution is due to the first nonzero term in the expansion of the initial datum as a series of eigenfunctions (1.11).

**Theorem 1.3.** *Let  $N = 3$ ,  $a \geq 0$ , and define  $H = -\Delta + \frac{a}{|x|^2}$ .*

- (i) *There exists  $C > 0$  such that, for all  $f \in L^2(\mathbb{R}^3)$  with  $|x|^{-\alpha_1} f \in L^1(\mathbb{R}^3)$ ,*

$$\| |x|^{\alpha_1} e^{-itH} f(\cdot) \|_{L^\infty} \leq C t^{-\frac{3}{2} + \alpha_1} \| |x|^{-\alpha_1} f \|_{L^1}.$$

- (ii) *For all  $k \in \mathbb{N}$ ,  $k \geq 1$ , there exists  $C_k > 0$  such that, for all  $f \in \mathcal{U}_k^\perp$  with  $|x|^{-\alpha_k} f \in L^1(\mathbb{R}^3)$ ,*

$$\| |x|^{\alpha_k} e^{-itH} f(\cdot) \|_{L^\infty} \leq C_k t^{-\frac{3}{2} + \alpha_k} \| |x|^{-\alpha_k} f \|_{L^1}.$$

The rest of the paper is devoted to the proof of Theorems 1.1 and 1.3. A final section is devoted to the description of the same phenomenon for the electromagnetic heat flow  $e^{-tH}$ , which enjoys the same scaling invariance of  $e^{-itH}$ .

## 2. PROOF OF THEOREMS 1.1 AND 1.3

The proof of Theorems 1.1 is constructive: assuming (1.10), we can construct an explicit initial datum  $u_0 \in L^1$  such that  $e^{-itH} u_0 \notin L^\infty$ . The argument strongly relies on the strategy which leads to the representation formula (1.6). In order to do this, we start with some preliminaries, concerning the functional setting of our problem. The following is analogous to Section 2 in [4]. We write it here for the sake of completeness (see also [7] for further details).

Define the following Hilbert spaces:

- the completion  $\mathcal{H}$  of  $C_c^\infty(\mathbb{R}^N \setminus \{0\}, \mathbb{C})$  with respect to the norm

$$\|\phi\|_{\mathcal{H}} = \left( \int_{\mathbb{R}^N} \left( |\nabla \phi(x)|^2 + \left( |x|^2 + \frac{1}{|x|^2} \right) |\phi(x)|^2 \right) dx \right)^{1/2};$$

- the completion  $\tilde{\mathcal{H}}$  of  $C_c^\infty(\mathbb{R}^N, \mathbb{C})$  with respect to the norm

$$\|\phi\|_{\tilde{\mathcal{H}}} = \left( \int_{\mathbb{R}^N} \left( |\nabla \phi(x)|^2 + (|x|^2 + 1) |\phi(x)|^2 \right) dx \right)^{1/2};$$

- the completion  $\mathcal{H}_A$  of  $C_c^\infty(\mathbb{R}^N \setminus \{0\}, \mathbb{C})$  with respect to the norm

$$\|\phi\|_{\mathcal{H}_A} = \left( \int_{\mathbb{R}^N} (|\nabla_{\mathbf{A}} \phi(x)|^2 + (|x|^2 + 1)|\phi(x)|^2) dx \right)^{1/2}$$

$$\text{with } \nabla_{\mathbf{A}} \phi = \nabla \phi + i \frac{\mathbf{A}(x/|x|)}{|x|} \phi.$$

It is clear that  $\mathcal{H} \hookrightarrow \tilde{\mathcal{H}}$ , with continuous embedding. This, together with [10, Proposition 6.1], gives in addition that  $\mathcal{H} \hookrightarrow L^p(\mathbb{R}^n)$ , with compact embedding, for all

$$2 \leq p < \begin{cases} 2^* = \frac{2N}{N-2}, & \text{if } N \geq 3, \\ +\infty, & \text{if } N = 2. \end{cases}$$

In analogy with [4], in the next result, by a pseudoconformal change of variables (see e.g. [10]), we reduce the hamiltonian  $H$  in equation (1.2) to a new operator, with an harmonic oscillator involved.

**Lemma 2.1.** *Let (1.5) hold and  $u = e^{-itH}u_0$ . Then*

$$(2.1) \quad \varphi(x, t) = (1 + t^2)^{\frac{N}{4}} u(\sqrt{1 + t^2}x, t) e^{-it\frac{|x|^2}{4}}$$

satisfies

$$\begin{aligned} \varphi &\in C(\mathbb{R}; L^2(\mathbb{R}^N)), \quad \varphi(x, 0) = u(x, 0), \\ \|\varphi(\cdot, t)\|_{L^2(\mathbb{R}^N)} &= \|u(\cdot, t)\|_{L^2(\mathbb{R}^N)} \text{ for all } t \in \mathbb{R}, \end{aligned}$$

and

$$(2.2) \quad i \frac{d\varphi}{dt}(x, t) = \frac{1}{(1 + t^2)} \left( H\varphi(x, t) + \frac{1}{4}|x|^2 \varphi(x, t) \right).$$

Denote now by

$$(2.3) \quad T : \mathcal{H} \rightarrow \mathcal{H}^*, \quad T = H + \frac{1}{4}|x|^2,$$

naturally defined via the associated (positive) quadratic form. Assumption (1.5) gives

$$(2.4) \quad \int_{\mathbb{R}^N} \left[ |\nabla_{\mathbf{A}} \phi(x)|^2 - \frac{a(x/|x|)}{|x|^2} |\phi(x)|^2 + \frac{|x|^2}{4} |\phi(x)|^2 \right] dx \geq C(N, \mathbf{A}, a) \|\phi\|_{\mathcal{H}}^2,$$

for some  $C = C(N, \mathbf{A}, a) > 0$  and for all  $\phi \in \mathcal{H}$ .

The following proposition, which describes completely the spectrum of  $T$ , was proved in [4, Proposition 3.2].

**Proposition 2.2.** *Let  $\mathbf{A} \in C^1(\mathbb{S}^{N-1}, \mathbb{R}^N)$  and  $a \in L^\infty(\mathbb{S}^{N-1})$ , and assume (1.5). Then*

$$\sigma(T) = \sigma_p(T) = \{\gamma_{m,k} : k, m \in \mathbb{N}, k \geq 1\}$$

where

$$(2.5) \quad \gamma_{m,k} = 2m - \alpha_k + \frac{N}{2}, \quad \alpha_k = \frac{N-2}{2} - \sqrt{\left(\frac{N-2}{2}\right)^2 + \mu_k},$$

$\mu_k$  is the  $k$ -th eigenvalue of the operator  $L$  on  $L^2(\mathbb{S}^{N-1})$  and  $\alpha_j, \beta_j$  are defined in (1.8). Each eigenvalue  $\gamma_{m,k}$  has finite multiplicity equal to

$$\#\left\{j \in \mathbb{N}, j \geq 1 : \frac{\gamma_{m,k}}{2} + \frac{\alpha_j}{2} - \frac{N}{4} \in \mathbb{N}\right\}$$

and a basis of the corresponding eigenspace is

$$\left\{ V_{n,j} : j, n \in \mathbb{N}, j \geq 1, \gamma_{m,k} = 2n - \alpha_j + \frac{N}{2} \right\},$$

where  $V_{n,j}$  is defined in (1.11).

**Remark 2.3.** It is easy to check that

$$\text{if } (m_1, k_1) \neq (m_2, k_2) \quad \text{then} \quad V_{m_1, k_1} \text{ and } V_{m_2, k_2} \text{ are orthogonal in } L^2(\mathbb{R}^N).$$

By Proposition 2.2 and classical spectral theory, it follows that

$$\left\{ \tilde{V}_{n,j} = \frac{V_{n,j}}{\|V_{n,j}\|_{L^2(\mathbb{R}^N)}} : j, n \in \mathbb{N}, j \geq 1 \right\}$$

is an orthonormal basis of  $L^2(\mathbb{R}^N)$ .

Notice, in addition, that  $\tilde{V}_{n,j} \in L^1(\mathbb{R}^N)$ , since  $\alpha_j < N$ , for any  $j$ , by (1.8). In order to prove our main result, it is fundamental to study the solution of (1.2), with initial datum  $\tilde{V}_{n,j}$ .

**Theorem 2.4.** *Let  $a \in L^\infty(\mathbb{S}^{N-1}, \mathbb{R})$ ,  $\mathbf{A} \in C^1(\mathbb{S}^{N-1}, \mathbb{R}^N)$ , and assume (1.1) and (1.5). Then, for any  $n, j \in \mathbb{N}$ ,  $j \geq 1$ ,*

$$(2.6) \quad e^{-itH} \tilde{V}_{n,j}(x) = (1+t^2)^{-\frac{N}{4} + \frac{\alpha_j}{2}} |x|^{-\alpha_j} \frac{e^{\frac{-|x|^2}{4(1+t^2)}}}{\|V_{n,j}\|_{L^2(\mathbb{R}^N)}} e^{i\frac{|x|^2 t}{4(1+t^2)}} e^{-i\gamma_{n,j} \arctan t} \psi_j\left(\frac{x}{|x|}\right) P_{n,j}\left(\frac{|x|^2}{2(1+t^2)}\right).$$

*Proof.* Let  $u_0 = \tilde{V}_{n,j}$ . We notice that, since  $2\alpha_j < N$ ,  $u_0 \in L^2$ , hence we can expand it in Fourier series with respect to the orthonormal system  $\{\tilde{V}_{m,k} : m, k \in \mathbb{N}, k \geq 1\}$ , i.e.

$$(2.7) \quad u_0 = \sum_{\substack{m, k \in \mathbb{N} \\ k \geq 1}} c_{m,k} \tilde{V}_{m,k} \quad \text{in } L^2$$

with

$$c_{m,k} = \int_{\mathbb{R}^N} u_0(x) \overline{\tilde{V}_{m,k}(x)} dx = \begin{cases} 1, & \text{if } (m, k) = (n, j), \\ 0, & \text{otherwise.} \end{cases}$$

In a similar way, for  $t > 0$ , we can expand the function  $\varphi(\cdot, t)$  defined in (2.1) (with  $u = e^{-itH} u_0$ ) as

$$(2.8) \quad \varphi(\cdot, t) = \sum_{\substack{m, k \in \mathbb{N} \\ k \geq 1}} \varphi_{m,k}(t) \tilde{V}_{m,k} \quad \text{in } L^2(\mathbb{R}^N),$$

where

$$\varphi_{m,k}(t) = \int_{\mathbb{R}^N} \varphi(x, t) \overline{\tilde{V}_{m,k}(x)} dx.$$

Since  $\varphi(z, t)$  satisfies (2.2), we obtain that  $\varphi_{m,k} \in C^1(\mathbb{R}, \mathbb{C})$  and

$$i\varphi'_{m,k}(t) = \frac{\gamma_{m,k}}{1+t^2} \varphi_{m,k}(t), \quad \varphi_{m,k}(0) = c_{m,k},$$

which by integration yields  $\varphi_{m,k}(t) = c_{m,k}e^{-i\gamma_{m,k} \arctan t}$ . Hence expansion (2.8) can be rewritten as

$$\varphi(z, t) = \sum_{\substack{m,k \in \mathbb{N} \\ k \geq 1}} c_{m,k} e^{-i\gamma_{m,k} \arctan t} \tilde{V}_{m,k}(z) = e^{-i\gamma_{n,j} \arctan t} \tilde{V}_{n,j}(z),$$

for all  $t > 0$ . It follows by (2.1) that

$$\begin{aligned} e^{-itH} \tilde{V}_{n,j}(x) &= e^{i\frac{t|x|^2}{4(1+t^2)}} (1+t^2)^{-\frac{N}{4}} \varphi\left(\frac{x}{\sqrt{1+t^2}}, t\right) \\ &= (1+t^2)^{-\frac{N}{4} + \frac{\alpha_j}{2}} |x|^{-\alpha_j} \frac{e^{\frac{-|x|^2}{4(1+t^2)}}}{\|V_{n,j}\|_{L^2(\mathbb{R}^N)}} e^{i\frac{|x|^2 t}{4(1+t^2)}} e^{-i\gamma_{n,j} \arctan t} \psi_j\left(\frac{x}{|x|}\right) P_{n,j}\left(\frac{|x|^2}{2(1+t^2)}\right), \end{aligned}$$

and the proof is now complete.  $\square$

*Proof of Theorem 1.1.* The proof of Theorem 1.1 is now a straightforward consequence of the previous result. It is sufficient to notice that condition (1.10) implies that  $\alpha_1 > 0$ , by (1.8). Therefore, taking as initial datum  $u_0 = \tilde{V}_{0,1}$  for (1.2) and observing that  $P_{1,0} \equiv 1$ , formula (2.6) gives that  $e^{-itH} u_0 \notin L^\infty$ .  $\square$

**Remark 2.5.** Formula (2.6) also gives a quite precise description of the time-decay phenomenon of each Fourier mode. Denote by  $\pi_{n,j}$  the projector of  $L^2(\mathbb{R}^N)$  on the eigenspace generated by  $V_{n,j}$ , namely

$$\pi_{n,j} f := c_{n,j} \tilde{V}_{n,j}, \quad c_{n,j} := \int_{\mathbb{R}^N} f(x) \overline{\tilde{V}_{n,j}(x)} dx,$$

and assume that  $\mu_j \geq 0$ , which implies by (1.8) that  $\alpha_j \leq 0$ . Identity (2.6) hence easily gives

$$(2.9) \quad \| |x|^{-\gamma_j} e^{-itH} \pi_{n,j} u_0 \|_{L^\infty} \leq C |t|^{-\frac{N}{2} - \gamma_j} \| |x|^{\gamma_j} \pi_{n,j} u_0 \|_{L^1}, \quad \gamma_j := -\alpha_j,$$

for some  $C > 0$  independent on  $u_0$  (but depending on  $j$  and  $n$ ). In particular, in the weighted topologies  $L^1(|x|^{\gamma_j} dx)$ ,  $L^\infty(|x|^{-\gamma_j} dx)$ , the decay of the evolution  $j$ -th mode of the initial datum, with respect to  $V_{n,j}$ , is faster than the usual one by a polynomial factor  $\alpha_j$ . This perfectly matches the diamagnetic phenomenon already observed in [6, 8]. Notice that, in the free case  $\mathbf{A} \equiv a \equiv 0$ , we have  $\mu_1 = 0$  and the leading term for the time decay is given by the 0-th mode, which gives the usual decay rate  $|t|^{-\frac{N}{2}}$ . Nevertheless, proving a time-decay estimate for  $e^{-itH}$  is both a matter of zero modes, which can be rephrased by the condition  $\mu_1 \geq 0$ , and of high modes, which is concerned with the uniform convergence of a series with generic term as in (2.9).

In the case  $N = 3$ ,  $\mathbf{A} \equiv 0$  and  $a \geq 0$  in which the series in (1.7) is bounded and the classical time decay consequently holds, the improved time decay observed in (2.9) for initial data of type (1.11) can be extended to initial data which are orthogonal to the space generated by the first modes, as stated in Theorem 1.3.

*Proof of Theorem 1.3.* If the initial datum  $u_0$  in (1.2) belongs to  $\mathcal{U}_k^\perp$  (in the case  $k = 1$  if  $u_0 \in L^2(\mathbb{R}^3)$ ), it can be expanded in Fourier series as

$$u_0 = \sum_{\substack{m,j \in \mathbb{N} \\ j \geq k}} c_{m,j} \tilde{V}_{m,k} \quad \text{in } L^2(\mathbb{R}^3), \quad \text{where } c_{m,j} = \int_{\mathbb{R}^3} u_0(x) \overline{\tilde{V}_{m,j}(x)} dx.$$

Then all the series expansions appearing in the proof of [4, Theorem 1.3] start from the index  $k$  instead of 1. Therefore, if the initial datum  $u_0$  belongs to  $\mathcal{U}_k^\perp$ , the representation formula (1.6) can be refined as

$$(2.10) \quad u(x, t) = \frac{e^{\frac{i|x|^2}{4t}}}{i(2t)^{3/2}} \int_{\mathbb{R}^3} K_k\left(\frac{x}{\sqrt{2t}}, \frac{y}{\sqrt{2t}}\right) e^{i\frac{|y|^2}{4t}} u_0(y) dy,$$

where

$$K_k(x, y) = \sum_{j \geq k} i^{-\beta_k} j_{-\alpha_k}(|x||y|) \psi_k\left(\frac{x}{|x|}\right) \overline{\psi_k\left(\frac{y}{|y|}\right)}.$$

From [4, Proof of Theorem 1.1] it follows that

$$(2.11) \quad |K_k(x, y)| \leq C(|x||y|)^{-\alpha_k} \quad \text{if } |x||y| \leq \delta,$$

for some constant  $C > 0$  depending on  $\delta$  and  $k$  but independent of  $x, y$ ; furthermore from [4, estimate (6.16)] it follows that

$$(2.12) \quad \sup_{x, y \in \mathbb{R}^3} |K_k(x, y)| < +\infty.$$

Combining (2.11) and (2.12) and recalling that  $\alpha_k < 0$ , we conclude that

$$(2.13) \quad |K_k(x, y)| \leq \text{const} (|x||y|)^{-\alpha_k}$$

for all  $x, y \in \mathbb{R}^2$ , for some constant  $\text{const} > 0$  independent of  $x, y$ . The conclusion follows from estimate (2.13) and the representation formula (2.10).  $\square$

## APPENDIX A. HEAT SEMIGROUP

We conclude this note with a small appendix devoted to the heat semigroup  $e^{-tH}$ . Since it enjoys the same scaling invariance of  $e^{-itH}$ , it is natural to expect analogous phenomena occurring at the level of  $L^p - L^{p'}$  time-decay. Moreover, notice that by the Barry Simon's diamagnetic inequality

$$|e^{-tH} f| \leq |e^{t\Delta} f|, \quad \text{provided } a(\theta) \geq 0,$$

for all  $t > 0$  (see [18]). Hence we easily obtain that

$$\|e^{-tH} u_0\|_{L^\infty} \leq C|t|^{-\frac{3}{2}} \|u_0\|_{L^1},$$

for all  $t > 0$ , provided  $a(\theta) \geq 0$ , as a consequence of the same estimate for  $e^{-t\Delta}$ . In the general case (i.e. if  $a$  has any sign), the phenomenon of lack of the classical  $L^1 - L^\infty$  bound was completely described in [9], where sharp decay estimates of  $L^q$ -norms for nonnegative Schrödinger heat semigroups were established. In the spirit of Theorem 2.4, we construct below an explicit example confirming the lack of the classical  $L^1 - L^\infty$  bound proved in [9]. Here the situation is quite simpler, due to the following self-similarity issue. Let us consider the equation

$$(A.1) \quad v_t - \Delta v + a \frac{v}{|x|^2} = 0, \quad a > -\left(\frac{N-2}{2}\right)^2.$$

We look for solutions of the form

$$v(x, t) = t^{-\mu} \phi\left(\frac{r}{t^\nu}\right) \psi_k(\theta), \quad \text{where } x = r\theta, \ r \geq 0, \ \theta \in \mathbb{S}^{N-1},$$



and  $\psi_k$  is as in (1.4) with  $\mathbf{A} \equiv 0$  for some  $k \geq 1$ . By the change of variables  $s = \frac{r}{t^\nu}$  and  $\nu = \frac{1}{2}$ , we obtain

$$(A.2) \quad 0 = v_t - v'' - \left(\frac{N-1}{r}\right)v' + \frac{a}{r^2}v - \frac{1}{r^2}\Delta_{\mathbb{S}^{N-1}}v \\ \equiv -t^{-\mu-1}\psi_k(\theta)\left(\phi''(s) + \left(\frac{N-1}{s} + \frac{s}{2}\right)\phi'(s) + \left(\mu - \frac{\mu_k}{s^2}\right)\phi(s)\right),$$

where  $\mu_k$  is the  $k$ -th eigenvalue of problem (1.4). Denoting by  $\phi(s) = s^{-\alpha}e^{-\beta s^\gamma}$ , we now get

$$(A.3) \quad \begin{cases} \phi'(s) = \left(-\frac{\alpha}{s} - \gamma\beta s^{\gamma-1}\right)\phi(s), \\ \phi''(s) = \left(\frac{\alpha}{s^2} - \beta\gamma(\gamma-1)s^{\gamma-2} + \left(\frac{\alpha}{s} + \beta\gamma s^{\gamma-1}\right)^2\right)\phi(s). \end{cases}$$

By (A.2) and (A.3), with the choices  $\alpha = \alpha_k$  with  $\alpha_k$  as in (1.8),  $\beta = \frac{1}{4}$ ,  $\gamma = 2$  and  $\mu = \frac{N-\alpha_k}{2}$ , it follows that

$$v_t - v'' - \left(\frac{N-1}{r}\right)v' + \frac{a}{r^2}v - \frac{1}{r^2}\Delta_{\mathbb{S}^{N-1}}v = 0.$$

In conclusion,

$$v(x, t) = t^{-\frac{N}{2}+\alpha_k}|x|^{-\alpha_k}e^{-\frac{|x|^2}{4t}}\psi_k\left(\frac{x}{|x|}\right)$$

is a solution to (A.1), with initial datum  $v(x, t_0) \in L^1(\mathbb{R}^N)$  (with  $t_0 > 0$ ). Again we notice that condition (1.10), which now reads  $a < 0$ , implies that, taking  $k = 1$ ,  $v(\cdot, t) \notin L^\infty(\mathbb{R}^N)$ , for any  $t > 0$ , thus confirming the lack of the classical  $L^1 - L^\infty$  bound.

Furthermore, an improvement of the decay for higher positive modes holds also in this case; indeed, the above example shows that, for any  $M > 0$  there exists an initial datum  $u_0$  (it is enough to take above  $k$  large enough) for which the solution decays faster than  $|t|^{-M}$ .

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